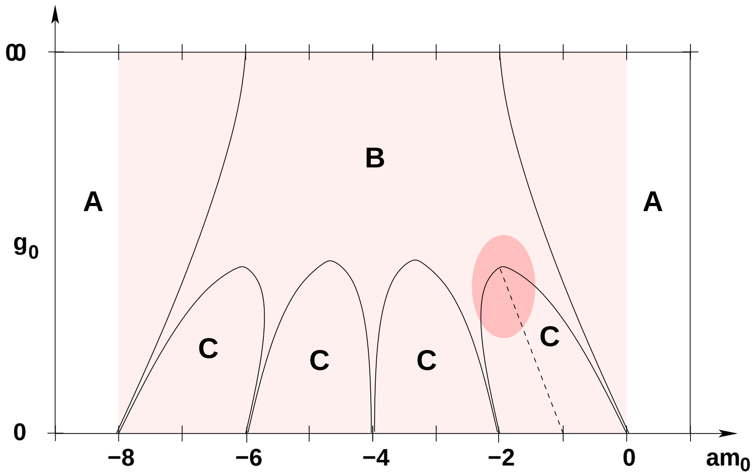
# Is there an Aoki phase in quenched QCD?

Maarten Golterman, Stephen R. Sharpe, Bob Singleton

# Outline

- Motivations and background
- Generalizing unquenched analysis [Sharpe & Singleton] to quenched theory
  - Field theoretic description of quenched Wilson fermions
  - Symmetries of quenched theory
  - Effective continuum Lagrangian (up to $O(a^2)$)
  - Effective chiral Lagrangian (up to $O(a^2)$)
  - Vacuum for $N_c \to \infty$
  - Generalization for $N_c = 3$?
- Summary

# Aoki phase with unquenched Wilson fermions




- Region B is Aoki phase
- $\langle i\bar{q}\gamma_5\tau_3 q\rangle \neq 0$
- Flavor and parity broken
- Massless Goldstone Bosons ($\pi^{\pm}$)
- Non-zero density of near-zero eigenmodes of $H_W = \gamma_5(D + m_0 + W)$:
- $\rho(0) \propto \langle i\bar{q}\gamma_5\tau_3 q\rangle \neq 0$

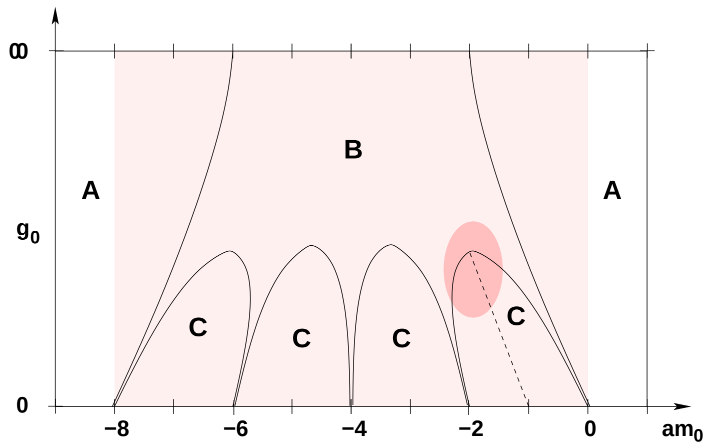
Can understand using Wilson $\chi$PT[Creutz,Sharpe & Singleton]

- Condensate swings from $\Sigma = 1$ ($m > 0$) to $\Sigma = -1$ ($m < 0$)
- Occurs when $m_{\rm phys} \sim a^2\Lambda^3_{\rm QCD}$

As $N_c \to \infty$, same analysis applies for a single flavor

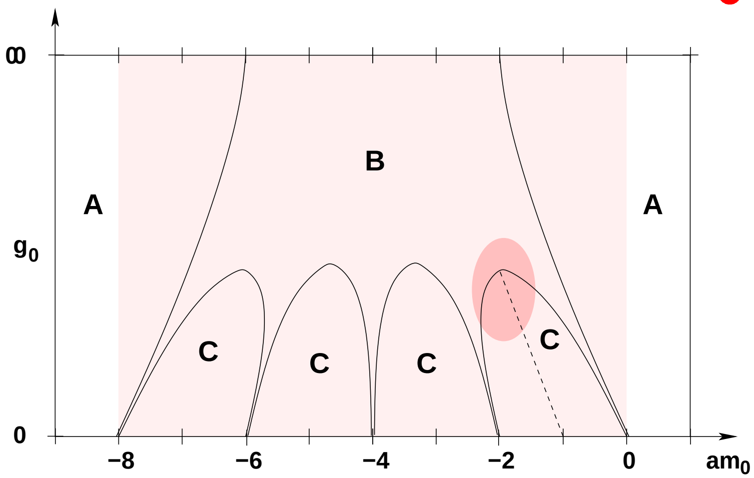
- $\rho(0) \propto \langle \bar{q}i\gamma_5 q\rangle \neq 0$
- **NO GB's SINCE NO CONTINUOUS LATTICE SYM. BROKEN**

# Why we care about the quenched Aoki phase




- Theoretical interest
- Explain results of numerical simulations
  - phase structure as in unquenched theory [Aoki *et al*, . . . ]
  - $\rho(0) \neq 0$ throughout supercritical region $-8 < m_0 a < 0$ [Edwards et al]
- DWF/Overlap use "quenched" $H_W(m_0 \sim -1)$ as a kernel
  - Density of extended near-zero modes leads to loss of chiral symmetry for DWF, and of locality for Overlap fermions [Golterman & Shamir]
  - $\Rightarrow$ need to be far away from any Aoki phase

# Conjecture for quenched QCD [Golterman & Shamir]




- BBN-like [Berruto *et al*] dislocations with size $\ell \sim a$ peppered throughout supercritical region ...
  - Explains $\rho(0) \neq 0$
  - Eigenmodes are **LOCALIZED** so DWF/Overlap OK
  - Goldstone's theorem evaded by $\langle P^+(x) P^-(y)\rangle \propto 1/m_{tw}$
- ...except if there is an Aoki phase in which:
  - $\rho(0) \neq 0$ due to delocalized modes: $\ell \geq \Lambda_{\mathrm{QCD}}^{-1}$
  - Goldstone's theorem satisfied in usual way
- Only expect $\chi$PT analysis to be sensitive to **long-distance** contributions to condensate and $\rho(0)$

# Step 1: define quenched theory

Lattice Lagrangian for quarks (we consider $N = 1$ or $2$ flavors)

$$\mathcal{L}_q = \bar{q}(D + M_0 + W)q$$

Following [Morel] add ghosts to cancel determinant

$$\mathcal{L}_g = \tilde{q}^\dagger(D + M_0 + W)\tilde{q}$$

Problem: In supercritical region $D + M_0 + W$ has **NEGATIVE** eigenvalues $\Rightarrow$ **Ghost functional integral not defined**

Solution: change variables before quenching (non-anomalous):

$$q = \exp(\pm i(\pi/4)\gamma_5)q' \,, \qquad \bar{q} = \bar{q}' \exp(\pm i(\pi/4)\gamma_5)$$

$$\bar{q}(D + M_0 + W)q \;=\; \bar{q}'(D \pm i\gamma_5[M_0 + W])q'$$

Fermion matrix now antihermitian, eigenvalues imaginary

# Step 1: continued

Rotated lattice Lagrangian for quarks

$$\mathcal{L}_q = \bar{q}'(D \pm i\gamma_5[M_0 + W])q'$$

Now add ghosts, with convergence term

$$\mathcal{L}_g = \tilde{q}^\dagger(D \pm i\gamma_5[M_0 + W])\tilde{q} + \epsilon\tilde{q}^\dagger\tilde{q}$$

- $\epsilon > 0$ regulates zero modes
- Assume can take $\epsilon \to 0$, since ignoring localized zero modes

Also include “convergence term” also in quark sector:

$$\Delta\mathcal{L}_q = \epsilon\bar{q}'q' = \mp\bar{q}i\gamma_5 q$$

- Selects direction of condensate in Aoki phase
- Direction can be chosen independently for each flavor

# Step 2: determine symmetry group [Damgaard *et al*]

Collect components into chiral "super fields"

$$\Psi_{L,R} = \begin{pmatrix} q_{L,R} \\ \tilde{q}_{L,R} \end{pmatrix}, \ \gamma_5 \Psi_{L,R} = \pm \Psi_{L,R}, \ \overline{\Psi}_{L,R} = \begin{pmatrix} \bar{q}_{L,R}, & \tilde{q}^{\dagger}_{R,L} \end{pmatrix}, \ (\overline{\Psi}_{L,R})\gamma_5 = \mp(\overline{\Psi}_{L,R})$$

Full Lagrangian in chiral form:

$$\mathcal{L}_W = \overline{\Psi}_L D \Psi_L + \overline{\Psi}_R D \Psi_R + \overline{\Psi}_R \underbrace{[\pm i(M_0 + W) - \epsilon]}_{\mathcal{M}} \Psi_L + \overline{\Psi}_L \underbrace{[\mp i(M_0 + W) - \epsilon]}_{\overline{\mathcal{M}}} \Psi_R$$

- $\overline{\Psi}_L|_g = \Psi_R|_g{}^{\dagger}$ required for convergence
- Symmetry group must maintain this relation for "bodies" of fields

Graded symmetry group ($N$ is number of flavors)

$$\Psi_{L,R} \to \mathcal{V}_{L,R}\Psi_{L,R}, \ \overline{\Psi}_{L,R} \to \overline{\Psi}_{L,R}\mathcal{V}^{-1}_{L,R}, \ \mathcal{M} \to \mathcal{V}_R \mathcal{M} \mathcal{V}_L^{-1}, \ \overline{\mathcal{M}} \to \mathcal{V}_L \overline{\mathcal{M}} \mathcal{V}_R^{-1}$$

- $\mathcal{G} = \{(\mathcal{V}_L, \mathcal{V}_R) \in [SL(N|N)_L \times SL(N|N)_R] \ltimes U(1)_V | \ \mathcal{V}_{Lgg}|_{\rm body} = \mathcal{V}^{\dagger -1}_{Rgg}|_{\rm body}\}$ .

# Step 3: determine continuum $\mathcal{L}_{\rm eff}$ [Symanzik]

Enforce lattice symmetries (including discrete) $\Rightarrow$

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{glue} + \overline{\Psi}(D \pm i\gamma_5 m + \epsilon)\Psi \pm a\overline{\Psi}b_1 i\gamma_5 i\sigma_{\mu\nu}F_{\mu\nu}\Psi + O(a^2)\,,$$

- Physical mass $m = Z_m(M_0 - m_c)$
- $\epsilon \to Z_P \epsilon$
- Unknown coefficient $b_1$
- Same form as for unquenched Wilson fermions except for $\pm i\gamma_5$ due to axial rotation
- $O(a^2)$ terms break no further symmetries
- $D \pm i\gamma_5 m \mp \gamma_5\sigma_{\mu\nu}F_{\mu\nu}$ is antihermitian, like the underlying lattice operator

# Step 4: determine chiral $\mathcal{L}_{\rm eff}$

Assume (based on numerical evidence) a non-vanishing (long distance) condensate, with corresponding Goldstone fields:

$$\Sigma \sim \langle \Psi_L \overline{\Psi}_R \rangle \longrightarrow \mathcal{V}_L \Sigma \mathcal{V}_R^{-1}, \quad \Sigma^{-1} \sim \langle \Psi_R \overline{\Psi}_L \rangle \longrightarrow \mathcal{V}_R \Sigma \mathcal{V}_L^{-1}$$

- Since quenched must keep singlet part $\Phi_0$

$$\Sigma = \exp(\Phi), \; \Phi_0 = -i \,\mathrm{str}\,(\Phi)$$

- Symmetry group is $\mathcal{G}$, but transformations enlarged to:

$$\mathcal{G}' = \{(\mathcal{V}_L, \mathcal{V}_R) \in GL(N|N)_L \times GL(N|N)_R \mid \mathcal{V}_{Lgg}|_{\rm body} = \mathcal{V}_{Rgg}^{\dagger -1}|_{\rm body}\},$$

- Thus $\Sigma \in GL(N|N)$ with contraint:

$$\Phi = \begin{pmatrix} i\phi_1 + \phi_2 & \overline{\chi} \\ \chi & \hat{\phi} \end{pmatrix}, \quad \hat{\phi}^\dagger|_{\rm body} = \hat{\phi}|_{\rm body}$$

- Keep only $\phi_1$ in quark sector (called $\phi$ below)

# Step 4: (continued)

Treat $\mathcal{M}, \overline{\mathcal{M}}$ as spurions, then set $\overline{\mathcal{M}} = \mathcal{M}^\dagger = \pm im + \epsilon$:

$$\begin{aligned}
\mathcal{L}_\chi \quad = \quad & \frac{f^2}{8} V_1(\Phi_0^2)\mathrm{str}\left(\partial_\mu\Sigma\partial_\mu\Sigma^{-1}\right) + \frac{c_0}{2}\Phi_0^2 \\
& \mp ic_1 \mathrm{str}\left(\Sigma - \Sigma^{-1}\right) - \epsilon\,\mathrm{str}\left(\Sigma + \Sigma^{-1}\right) \\
& + c_2\left[(\mathrm{str}\Sigma)^2 + \left(\mathrm{str}\Sigma^{-1}\right)^2\right] + c_3 \mathrm{str}\Sigma\,\mathrm{str}\Sigma^{-1} + c_4\left[\mathrm{str}\left(\Sigma^2\right) + \mathrm{str}\left(\Sigma^{-2}\right)\right]
\end{aligned}$$

- Keep up to quadratic order in $a$ and $m$ in potential
  - $f, c_0 \sim \Lambda_{\mathrm{QCD}}$
  - $c_1 \sim m + a$
  - $c_{2,3,4} \sim m^2 + am + a^2$
- Quenching triples number of quadratic terms
- Quenching implies $c_i$ are functions of $\Phi_0^2$
- Potential simplifies in large $N_c$ limit:
  - $f^2, c_1, c_4 \propto N_c$
  - $c_{0,2,3} \propto 1$
  - Can drop $\Phi_0$ dependence of $c_i$

# Step 5: Aoki phase in large $N_c$ limit

- Useful limit since quark and ghosts decouple, and quark sector is unquenched.
- In quark sector, minimize potential

$$\mathcal{V}_q = \mp i c_1 \mathrm{str}\left(\Sigma - \Sigma^{-1}\right) - \epsilon\, \mathrm{str}\left(\Sigma + \Sigma^{-1}\right) + c_4 \left[\mathrm{str}\left(\Sigma^2\right) + \mathrm{str}\left(\Sigma^{-2}\right)\right]$$

  - Interesting regime: $c_1 \sim m + a = m' \sim c_4 \sim a^2 \gg \epsilon$
  - Consider $c_4 < 0$ since leads to Aoki phase.
- Single quark flavor: $\Sigma = \exp(i\phi)$

$$\mathcal{V}_\chi = \pm 2c_1 \sin\phi + 2c_4 \cos(2\phi) - 2\epsilon \cos\phi$$

  - $4|c_4| \le c_1 \Rightarrow \phi = \mp\pi/2$, $\Sigma = \mp i$
  - $c_1 \le -4|c_4| \Rightarrow \phi = \pm\pi/2$ or $\mp 3\pi/2$, $\Sigma = \pm i$
  - $-4|c_4| \le c_1 \le 4|c_4| \Rightarrow \phi$ interpolates through $\phi = 0$, $\Sigma = 1$ (direction determined by $\epsilon$ term)
- In original variables $q$, condensate interpolates from $+1$ to $-1$ through Aoki phase with $\pm\langle\overline{q} i\gamma_5 q\rangle > 0$ (*as expected*)
- Thus long-distance contribution to $\rho(0)$ non-zero, although **NO GB's**

# Step 5: ghosts in large $N_c$ limit

- Expect $\langle q\bar{q}\rangle = \langle \tilde{q}\tilde{q}^\dagger\rangle$ unless graded symmetry broken
- However $\Sigma_g = \exp(\hat\phi)$ with $\hat\phi$ real: how can $\Sigma_g = \Sigma_q$ when latter is complex?
- Technical problem: Potential in ghost sector is unbounded below, and complex

$$\mathcal{V}_g = \pm 2ic_1 \sinh\hat\phi - 2c_4\cosh(2\hat\phi) + 2\epsilon\cosh\hat\phi$$

- "Solutions":
  - Need higher order terms when $|\hat\phi| > 1$ (unlike for $\phi$), so *we do not know about boundedness*
  - Assume that effective theory is sensible and integral over $\hat\phi$ converges at $\hat\phi \to \pm\infty$
  - "Minimize potential" really means "find saddle points" in integral over $\hat\phi$
  - Need to deform contour into complex $\hat\phi$ plane
  - Choose saddle with *maximum* $\Re\mathcal{V}_g$ (minimum $|\exp(-\mathcal{V}_g)|$), that can join onto $\hat\phi \to \pm\infty$

# Step 5: large $N_c$ ghost (continued)

- "Potential" in ghost sector ($\Sigma_g = \exp(\hat\phi)$):

$$\mathcal{V}_g = \pm 2 i c_1 \sinh\hat\phi - 2c_4\cosh(2\hat\phi) + 2\epsilon\cosh\hat\phi$$

- Saddle point equation ($\epsilon \to 0$):

$$\pm i c_1 \cosh\hat\phi = 2c_4 \sinh(2\hat\phi)$$

- Saddles are at:
  - $\hat\phi_A = \mp i\pi/2, \pm 3i\pi/2, \ldots$ (note: imaginary)
  - $\hat\phi_B = \pm i\pi/2, \mp 3i\pi/2, \ldots$
  - $\hat\phi_C$, which, for $|c_1| < 4|c_4|$, interpolates between saddles $A$ and $B$, e.g. $\hat\phi_C = \pm i \sin^{-1}(c_1/4c_4)$
- Key result: maximizing $\Re V_g$ (including $\epsilon$) chooses saddle such that $\Sigma_g = \Sigma_q$
- Steepest descent from saddles is in direction of real $\hat\phi$
- $\Rightarrow$ **Do get Aoki phase in ghost sector**

# Final Step: What happens at $N_c = 3$?

- Consider one-flavor case (adequate for condensate since $N$ independent)
- Values of $c_1$ and $c_4$ depend on $N_c$
- Potential now couples quark and ghost sectors

$$\Delta\mathcal{V}_\chi = \frac{c_0}{2}\Phi_0^2 + c_2\left[(\mathrm{str}\Sigma)^2 + \left(\mathrm{str}\Sigma^{-1}\right)^2\right] + c_3\mathrm{str}\Sigma\,\mathrm{str}\Sigma^{-1}$$

  - $c_0 \sim \Lambda_{\mathrm{QCD}} \gg c_{1-4} \sim a^2$
- Does $c_0$ dominate at $N_c = 3$?
  - NO: in quenched $\chi$PT $\Phi_0^2$ term changes single to double poles, but does not shift masses
  - $\Rightarrow$ $\Phi_0$ should not affect choice of saddles
  - Expect same to hold for $c_2$ and $c_3$ terms, which have two straces
- Does this work mathematically? In part:
  - Previous saddles (which have $\Phi_0 = 0$) remain solutions
  - But all have $V_q + V_g = 0$, so how distinguish?

# Conclusions

- We have argued that there is an Aoki phase in the quenched theory, with properties similar to those in the unquenched theory

# Conclusions

- We have argued that there is an Aoki phase in the quenched theory, with properties similar to those in the unquenched theory
- In quenched theory with $N = 2$, form of condensate depends on source. Can have (in terms of the original lattice fields before axial rotation)
  - $\langle \overline{q} i \gamma_5 \tau_3 q \rangle \neq 0$ with 2 Goldstone bosons
  - $\langle \overline{q} i \gamma_5 q \rangle \neq 0$ with NO Goldstone bosons (no flavor breaking)

# Conclusions

- We have argued that there is an Aoki phase in the quenched theory, with properties similar to those in the unquenched theory
- In quenched theory with $N = 2$, form of condensate depends on source. Can have (in terms of the original lattice fields before axial rotation)
  - $\langle \overline{q} i \gamma_5 \tau_3 q \rangle \neq 0$ with 2 Goldstone bosons
  - $\langle \overline{q} i \gamma_5 q \rangle \neq 0$ with NO Goldstone bosons (no flavor breaking)
- The same two options are also available in unquenched theory at large $N_c$, although for $N_c$ non-infinite flavor breaking condensate is favored

# Conclusions

- We have argued that there is an Aoki phase in the quenched theory, with properties similar to those in the unquenched theory
- In quenched theory with $N = 2$, form of condensate depends on source. Can have (in terms of the original lattice fields before axial rotation)
  - $\langle \overline{q} i \gamma_5 \tau_3 q \rangle \neq 0$ with 2 Goldstone bosons
  - $\langle \overline{q} i \gamma_5 q \rangle \neq 0$ with NO Goldstone bosons (no flavor breaking)
- The same two options are also available in unquenched theory at large $N_c$, although for $N_c$ non-infinite flavor breaking condensate is favored
- DWF/Overlap kernels need to take care to avoid the Aoki phases

# Conclusions

- We have argued that there is an Aoki phase in the quenched theory, with properties similar to those in the unquenched theory
- In quenched theory with $N = 2$, form of condensate depends on source. Can have (in terms of the original lattice fields before axial rotation)
  - $\langle \overline{q} i \gamma_5 \tau_3 q \rangle \neq 0$ with 2 Goldstone bosons
  - $\langle \overline{q} i \gamma_5 q \rangle \neq 0$ with NO Goldstone bosons (no flavor breaking)
- The same two options are also available in unquenched theory at large $N_c$, although for $N_c$ non-infinite flavor breaking condensate is favored
- DWF/Overlap kernels need to take care to avoid the Aoki phases
- Would be nice to tighten up parts of the argument . . . !